International Journal of Engineering, Management, Humanities and Social Sciences Paradigms
Volume 35, Issue 04 and Publication Date: 20th June, 2023
An Indexed, Referred and Peer Reviewed Journal
ISSN (Online): 2347-601X
www.ijemhs.com

Weighted Information divergence measures from a new series and their characteristics

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Abstract

As a family of Csiszar's functional divergence, new series of weighted divergence measures are introduced in this study. We also describe the properties of convex functions and weighted divergences, compare a number of weighted divergences, and derive numerous intriguing and significant relationships between weighted divergences of these new series and those of other well-known divergence measures.

Keywords: "New series of weighted information divergences, various relations among weighted divergences, comparison of divergences, bounds, mutual information"

1 Introduction

Divergence measures are essentially measures of the separation or comparison of two probability distributions. It implies that any deviation. The minimum value for the measure must be zero in the case of equal probability distributions and the maximum value in the case of perpendicular probability distributions. Several divergence measures are appropriate depending on the problem's nature. Thus, creating a new divergence measure is always beneficial.

Many studies on information divergence measures have been conducted recently by "Dragomir [9,10,11,12], Jain [15,16,19,20,21,22,24], Taneja [39,40,43,44,45]", and others. These studies provided the concept of information divergence measures as well as information on their characteristics, bounds and connections to other measures.

We have limited ourselves to weighted discrete probability distributions without losing fundamental knowledge., so let

$\Gamma_n = \{P = (p_1, p_2, \ldots, p_n), p_i > 0, \quad \sum_{i=1}^{n} p_i = 1, n \geq 2$

be the collection of all discrete probability distributions that are finite. Only for convenience, discrete distributions are limited in this case; identical findings apply to continuous distributions as well. If we take $p_i \geq 0$ for some i = 1to n ,then we have to suppose that 0 f (0) = $0f\left(\frac{0}{0}\right), \sum_{i=1}^{n} w_i p_i = 1$.

There have been several generalized functional information divergence measures introduced, characterized, and applied in a variety of fields, including: "Csiszar's f- divergence [6,7], Bregman's f- divergence [2], Burbea- Rao's f - divergence [3], Renyi's like f- divergence [35], and Jain Saraswat f- divergence [23]".

These generalized f- measures can be used to create a variety of divergence measures by accurately defining the function f. Due to its compact nature, provided by Csiszar's f- divergence, $C_f(P;Q) = \sum_{i=1}^{n} q_i f\left(\frac{p_i}{q_i}\right)$

And its weighted form is defined as given below by many authors as

$$C_f(P;Q;W) = \sum_{i=1}^{n} w_i q_i f\left(\frac{p_i}{q_i}\right), \tag{1.1}$$

where $P, Q \in \Gamma_n$.

Where f: $[0, \infty) \times (0, \infty) \to \mathcal{R}$ is real, continuous, and convex function & $P = (p_1, p_2, \ldots, p_n), p_i > 0, Q = (q_1, q_2, \ldots, q_n), q_i > 0, \in \Gamma_n$, where p_i and q_i are probabilities. Following are a few divergences caused by $C_f(P;Q;W)$,are as follows:

$$E_m^*(P;Q;W) = \sum_{i=1}^{n} w_i \frac{(p_i - q_i)^{2m}}{(p_i q_i)^{\frac{(2m-1)}{2}}} \text{, m = 1,2,3,...[24]} \tag{1.2}$$

$$J_m^*(P;Q;W) = \sum_{i=1}^{n} w_i \frac{(p_i - q_i)^{2m}}{(p_i q_i)^{\frac{(2m-1)}{2}}} \exp \frac{(p_i - q_i)^2}{p_i q_i} \text{, m = 1,2, 3,....[24]} \tag{1.3}$$

$$N_m^*(P;Q;W) = \sum_{i=1}^{n} w_i \frac{(p_i - q_i)^{2m}}{(p_i + q_i)^{2m-1}} \exp \frac{(p_i - q_i)^2}{(p_i + q_i)^2} \text{, m = 1,2,3,... [22]} \tag{1.4}$$

$$P^*(P;Q;W) = \sum_{i=1}^{n} w_i \frac{(p_i - q_i)^4 (p_i^2 + q_i^2)(p_i + q_i)}{p_i^3 q_i^3} \quad [21] \tag{1.5}$$

$$\Delta_{\mathrm{m}}(Ᵽ;Q;W)=\sum_{i=1}^{n} w_i \frac{(þ_i-q_i)^{2m}}{(þ_i+q_i)^{2m-1}} \quad , \mathrm{m} = 1,2,3,... \tag{1.6}$$

= weighted Puri and Vineze Divergences [28]

$$\chi^{2m}(Ᵽ;Q;W)=\sum_{i=1}^{n} w_i \frac{(þ_i-q_i)^{2m}}{q_i^{2m-1}} \quad , \mathrm{m} = 1,2,3,... \tag{1.7}$$

= weighted Chi- m divergences [49],

$$\mathrm{E}_1^*(Ᵽ;Q;W)=E^*(Ᵽ;Q;W)=\sum_{i=1}^{n} w_i \frac{(þ_i-q_i)^{2}}{\sqrt{þ_i q_i}} \tag{1.8}$$

$$\Delta_1(Ᵽ;Q;W)=\Delta(Ᵽ;Q;W)=\sum_{i=1}^{n} w_i \frac{(þ_i-q_i)^{2}}{(þ_i+q_i)} \tag{1.9}$$

= Weighted Triangular discrimination [8],

&

$$\chi^{2}(Ᵽ;Q;W)=\sum_{i=1}^{n} w_i \frac{(þ_i-q_i)^{2}}{q_i} \tag{1.10}$$

= weighted Chi- square divergence [33].

(1.8), (1.9), and (1.10) are the particular cases of (1.2), (1.6), & (1.7) respectively at m = 1.

$$K(Ᵽ;Q;W)=\sum_{i=1}^{n} w_i\, þ_i \log\left(\frac{þ_i}{q_i}\right)$$ = weighted Relative information [31].

(1.11)

$$G(Ᵽ;Q;W)=\sum_{i=1}^{n} w_i \left(\frac{þ_i+q_i}{2}\right) \log\left(\frac{þ_i+q_i}{2p_i}\right) \tag{1.12}$$

= weighted Relative Arithmetic- Geometric Divergence [43].

$$F(Ᵽ;Q;W)=\sum_{i=1}^{n} w_i\, þ_i \log\left(\frac{2þ_i}{þ_i+q_i}\right) \tag{1.13}$$

= weighted Relative Jensen- Shannon divergence [38]

There are a few means, which are as follows [(1.14)- (1.20)], that can be found in literature [42].

$$H^*(Ᵽ;Q;W)=\sum_{i=1}^{n} 2\frac{w_i þ_i q_i}{p_i+q_i}$$ = Weighted Harmonic mean divergence (1.14)

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$A(P;Q;W) = \sum_{i=1}^{n} w_i \left(\frac{p_i+q_i}{2}\right)$ = Weighted Arithmetic mean divergence (1.15)

$N_1(P;Q;W) = \sum_{i=1}^{n} w_i \left(\frac{\sqrt{p_i}+\sqrt{q_i}}{2}\right)^2$ = weighted Square root mean (1.16)

$N_3(P;Q;W) = \sum_{i=1}^{n} w_i \frac{(p_i+\sqrt{p_i q_i}+q_i)}{3}$ = weighted Heronian mean

(1.17)

$L^*(P;Q;W) = \sum_{i=1}^{n} w_i \frac{p_i-q_i}{\log p_i - \log q_i}$, p_i is not equal to q_i, for all i (1.18)

= weighted Logarithmic mean

$G^*(P;Q;W) = \sum_{i=1}^{n} w_i (p_i q_i)^{1/2}$ = Weighted Geometric mean (1.19)

$N_2(P;Q;W) = \sum_{i=1}^{n} w_i \left(\frac{\sqrt{p_i}+\sqrt{q_i}}{2}\right)\left(\sqrt{\frac{p_i+q_i}{2}}\right)$ = weighted N_2 mean (1.20)

$J_R(P;Q;W) = 2[F(P;Q;W) + G(P;Q;W)] = \sum_{i=1}^{n} w_i (p_i - q_i) \log\left(\frac{p_i+q_i}{2q_i}\right)$

(1.21)

= Relative weighted J- Divergence [11],

where F $(P;Q;W)$ and G$(P;Q;W)$ are given by (1.13) and (1.12) respectively.

$h(P;Q;W) = 1 - G^*(P;Q;W) = \frac{1}{2}\sum_{i=1}^{n} w_i \left(\sqrt{p_i} - \sqrt{q_i}\right)^2$ (1.22)

= “Weighted Hellinger Discrimination”

where $G^*(P;Q;W)$ is given by (1.19)

$$I(P;Q;W) = \frac{1}{2}[F(P;Q;W) + F(Q;P;W)]$$

$$= \frac{1}{2}\left[\sum_{i=1}^{n} w_i p_i \log\left(\frac{2p_i}{p_i+q_i}\right) + \sum_{i=1}^{n} w_i q_i \log\left(\frac{2q_i}{p_i+q_i}\right)\right] \quad (1.23)$$

= weighted JS divergence [3,38], where $F(P;Q;W)$ is given by (1.13)

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$$J(P;Q;W) = K((P;Q;W) + K(Q;P;W) = J_R(P;Q;W) + J_R(Q;P;W) = \sum_{i=1}^{n} w_i\,(p_i - q_i)\, log\frac{p_i}{q_i} \tag{1.24}$$

= weighted J- divergence [25,31],

where $J_R(P;Q;W)$ and $K((P;Q;W)$ are given by (1.21) and (1.11) respectively.

$$T(P;Q;W) = \frac{1}{2}[G(P;Q;W) + G(Q;P;W)]$$

$$= \frac{1}{2}\left[\sum_{i=1}^{n} w_i\left(\frac{p_i+q_i}{2}\right) log\left(\frac{p_i+q_i}{2\sqrt{p_i q_i}}\right)\right] \tag{1.25}$$

= weighted AG Mean Divergence [43]

where $G(P;Q;W)$ is given by (1.12).

$$\psi(P;Q;W) = \chi^2(P;Q;W) + \chi^2(Q;P;W) = \sum_{i=1}^{n} w_i \frac{(p_i+q_i)(p_i-q_i)^2}{(p_i q_i)} \tag{1.26}$$

= Weighted Symmetric Chi- square Divergence [12], where $\chi^2(P;Q;W)$ is given by (1.10). These all are with the condition with condition $\sum_{i=1}^{n} w_i\, p_i = 1$ and $\sum_{i=1}^{n} w_i\, q_i = 1$.

Divergences (1.2) to (1.4), (1.6), and (1.7) refer to a group of divergence measures that correspond to a group of convex functions. Jain and others present divergences (1.2) to (1.4) among them. With regard to probability distributions $P, Q \in \Gamma_n$. Divergences (1.2) to (1.6), Means (1.14) to (1.20), and (1.22) to (1.26) are symmetric while (1.7), (1.11) to (1.13), and (1.21) are non-symmetric.

Now for differential function f: $[0, \infty) \times (0, \infty) \to \mathscr{R}$.

Think about the related function

k: $[0, \infty) \times (0, \infty) \to \mathscr{R}$ where $k(\varepsilon, w) = w(\varepsilon - 1) f'\left(\frac{\varepsilon+1}{2}\right)$ (1.27)

After putting (1.27) in (1.1), we get

$$E^*{}_{C_f}(P;Q;W) = \sum_{i=1}^{n} w_i\,(p_i - q_i)\, f'\left(\frac{(p_i+q_i)}{2q_i}\right) \tag{1.28}$$

2 Brand-new convex functions and characteristics

Here, we create a few fresh convex function series and investigate their characteristics. Let f: [0, ∞) × (0, ∞) → ℛ be a mapping defined as the first step towards achieving this.

$$\mathrm{f_m}(\varepsilon, w) = \frac{w(\varepsilon^2-1)^{2m}}{\varepsilon^{2m-1}} \text{ , m = 1,2, 3, …, w} > 0 \qquad (2.1)$$

&

$$\mathrm{f}'_m(\varepsilon; w) = \frac{w(\varepsilon^2-1)^{2m-1}[\varepsilon^2(2m+1)+2m-1]}{\varepsilon^{2m}} \qquad (2.2)$$

$$\mathrm{f}''_m(\varepsilon; w) = \frac{w2m(\varepsilon^2-1)^{2m-2}[\varepsilon^4(2m+1)+4\varepsilon^2(m-1)+2m-1]}{\varepsilon^{2m+1}} \qquad (2.3)$$

From (2.1), we get the following new convex functions at m = 1,2,3... respectively.

$$\mathrm{f_1}(\varepsilon, w) = \frac{w(\varepsilon^2-1)^2}{\varepsilon}, \mathrm{f_2}(\varepsilon, w) = \frac{w(\varepsilon^2-1)^4}{\varepsilon^3}, \mathrm{f_3}(\varepsilon, w) = \frac{w(\varepsilon^2-1)^6}{\varepsilon^5}, \ldots\ldots\ldots \qquad (2.4)$$

As we are aware, a convex function also exists when linear combinations of convex functions are used.

(i) Take

$$\mathrm{f_{1,2}}(\varepsilon, w) = \mathrm{f_1}(\varepsilon, w) + \mathrm{f_2}(\varepsilon, w) = \frac{w(\varepsilon^2-1)^2[\varepsilon^4-\varepsilon^2+1)]}{\varepsilon^3} \qquad (2.5)$$

$$\mathrm{f_{2,3}}(\varepsilon, w) = \mathrm{f_2}(\varepsilon, w) + \mathrm{f_3}(\varepsilon, w) = \frac{w(\varepsilon^2-1)^4[\varepsilon^4-\varepsilon^2+1]}{\varepsilon^5} \qquad (2.6)$$

This allows us to write for m = 1, 2, 3...

$$\mathrm{f_{m,m+1}}(\varepsilon, w) = \mathrm{f_m}(\varepsilon, w) + \mathrm{f_{m+1}}(\varepsilon, w) = \frac{w(\varepsilon^2-1)^{2m}[\varepsilon^4-\varepsilon^2+1]}{\varepsilon^{2m+1}} \qquad (2.7)$$

(ii) Take

$$\mathrm{k_1}(\varepsilon, w) = \mathrm{f_1}(\varepsilon, w) + (\log_e b)\,\mathrm{f_2}(\varepsilon, w) + \frac{(\log_e b)^2}{2!}\mathrm{f_3}(\varepsilon, w) + \ldots\ldots\ldots\ldots$$

$$= \frac{w(\varepsilon^2-1)^2}{\varepsilon} + (\log_e b)\frac{w(\varepsilon^2-1)^4}{\varepsilon^3} + \ldots\ldots\ldots$$

$$= \frac{w(\varepsilon^2-1)^2}{\varepsilon}\left[1 + (\log_e b)\frac{(\varepsilon^2-1)^2}{\varepsilon^2} + \cdots\right] \qquad (2.8)$$

$$= \frac{w(\varepsilon^2-1)^2}{\varepsilon} b^{\frac{(\varepsilon^2-1)^2}{\varepsilon^2}} \text{ , b} > 1$$

Similarly

$$k_2(\varepsilon, w) = f_2(\varepsilon, w) + (\log_e b)\, f_3(\varepsilon, w) + \frac{(\log_e b)^2}{2!} f_4(\varepsilon, w) + \ldots\ldots\ldots\ldots$$

$$= \frac{w(\varepsilon^2-1)^4}{\varepsilon^3} + (\log_e b)\frac{w(\varepsilon^2-1)^6}{\varepsilon^5} + \ldots\ldots\ldots\ldots$$

$$= \frac{w(\varepsilon^2-1)^4}{\varepsilon^3}\left[1 + (\log_e b)\frac{(\varepsilon^2-1)^2}{\varepsilon^2} + \cdots\right] \tag{2.9}$$

$$= \frac{w(\varepsilon^2-1)^4}{\varepsilon^3}\, b^{\frac{(\varepsilon^2-1)^2}{\varepsilon^2}}, \; b > 1$$

So, we may take

$$k_m(\varepsilon, w) = \frac{w(\varepsilon^2-1)^{2m}}{\varepsilon^{2m-1}}\, b^{\frac{(\varepsilon^2-1)^2}{\varepsilon^2}}, \; b > 1, \; m = 1,2,3, \ldots, w > 0 \tag{2.10}$$

If we take b = e ≈ 2.71828, then we have

$$k_m(\varepsilon, w) = \frac{w(\varepsilon^2-1)^{2m}}{\varepsilon^{2m-1}}\, e^{\frac{(\varepsilon^2-1)^2}{\varepsilon^2}}, \; m = 1,2,3, \ldots., w > 0 \tag{2.11}$$

The following are characteristics of the functions defined by (2.1), (2.7) and (2.11).

$f_m(1,1) = 0 = f_{m,m+1}(1,1) = k_m(1,1)$ this implies $f_m(\varepsilon, w)$, $f_{m,m+1}(\varepsilon, w)$ & $k_m(\varepsilon, w)$ are normal for each m. Also $f''_m(\varepsilon, w) \geq 0$ for all $\varepsilon \in (0, \infty), w > 0$ hence $f_m(\varepsilon, w)$ is convex function & so $f_{m,m+1}(\varepsilon, w)$ & $k_m(\varepsilon, w)$ also convex.

Also $f'_m(\varepsilon, w) < 0$ at (0,1) & > 0 at (1,∞) and w> 0, thus $f_m(\varepsilon, w)$is decreasing at (0,1) & increasing at (1,∞), for every value of m, and $f'_m(1,1) = 0$.

3 New series of information weighted divergence measures and properties:

The new series of weighted divergence measures obtained in this part are studied for their characteristics as they relate to the new series of convex functions developed in section 2 because of the theorem given below is first and is popular in the literary world [7].

Theorem 3.1 Let function f is convex and normalize., $f''(\varepsilon, w) \geq 0 \; \forall \; \varepsilon > 0$, w>0 and f (1,1) = 0 respectively, then $C_f(\text{Ᵽ}; \text{Ọ}; W)$and its adjoint $C_f(\text{Ọ}; \text{Ᵽ}; W)$are convex and non-negative in the pair of probability distribution Ᵽ, Ọ $\in \Gamma_n \times \Gamma_n$

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Now put (2.1) into (1.1), and the new weighted divergences that result are as follows.

$$C_f(P;Q;W) = \Upsilon_m(P;Q;W) = \sum_{i=1}^{n} w_i \frac{(p_i^2-q_i^2)^{2m}}{p_i^{2m-1} q_i^{2m}} \quad , \mathrm{m} = 1,2,3,\ldots, \mathrm{w} > 0 \tag{3.1}$$

$$\Upsilon_1(P;Q;W) = \sum_{i=1}^{n} w_i \frac{(p_i^2-q_i^2)^2}{p_i q_i^2} \quad , \quad \Upsilon_2(P;Q;W) = \sum_{i=1}^{n} w_i \frac{(p_i^2-q_i^2)^4}{p_i^3 q_i^4}, \ldots \tag{3.2}$$

Similarly put (2.7) into (1.1), and the new weighted divergences that result are as follows.

$$C_f(P;Q;W) = \eta_m(P;Q;W) = \sum_{i=1}^{n} w_i \frac{(p_i^2-q_i^2)^{2m}(p_i^4-p_i^2 q_i^2+q_i^4)}{p_i^{2m+1} q_i^{2m+2}} \quad , \mathrm{m} = 1,2,3,\ldots \tag{3.3}$$

$$\eta_1(P;Q;W) = \sum_{i=1}^{n} w_i \frac{(p_i^2-q_i^2)^2(p_i^4-p_i^2 q_i^2+q_i^4)}{p_i^3 q_i^4} \tag{3.4}$$

$$\eta_2(P;Q;W) = \sum_{i=1}^{n} w_i \frac{(p_i^2-q_i^2)^4(p_i^4-p_i^2 q_i^2+q_i^4)}{p_i^5 q_i^6} \tag{3.5}$$

Similarly put (2.11) into (1.1), and the new weighted divergences that result are as follows.

$$C_f(P;Q;W) = \rho_m(P;Q;W) = \sum_{i=1}^{n} w_i \frac{(p_i^2-q_i^2)^{2m}}{p_i^{2m-1} q_i^{2m}} \exp \frac{(p_i^2-q_i^2)^2}{(p_i q_i)^2} \quad , \mathrm{m} = 1,2,3,\ldots \tag{3.6}$$

$$\rho_1(P;Q;W) = \sum_{i=1}^{n} w_i \frac{(p_i^2-q_i^2)^2}{p_i q_i^2} \exp \frac{(p_i^2-q_i^2)^2}{(p_i q_i)^2} \tag{3.7}$$

$$\rho_2(P;Q;W) = \sum_{i=1}^{n} w_i \frac{(p_i^2-q_i^2)^4}{p_i^3 q_i^4} \exp \frac{(p_i^2-q_i^2)^2}{(p_i q_i)^2} \tag{3.8}$$

The following are the characteristics of divergences as specified by (3.1), (3.3) and (3.6).

- Theorem 3.1 allows us to state that $\Upsilon_m(P;Q;W)$, $\eta_m(P;Q;W)$, $\rho_m(P;Q;W) > 0$ and are convex in the pair of probability distribution

$P, Q \in \Gamma_n$.

• $\Upsilon_m(Ᵽ; Ọ; W) = 0 = \eta_m(Ᵽ; Ọ; W) = \rho_m(Ᵽ; Ọ; W)$ if $Ᵽ = Ọ = W$ or $ƥ_i = q_i = w_i$ (attains its minimum value).

• Since $\Upsilon_m(Ᵽ; Ọ; W) \neq \Upsilon_m(Ọ; Ᵽ; W), \eta_m(Ᵽ; Ọ; W) \neq \eta_m(Ọ; Ᵽ; W)$, $\rho_m(Ᵽ; Ọ; W) \neq \rho_m(Ọ; Ᵽ; W) \Rightarrow \Upsilon_m(Ᵽ; Ọ; W), \eta_m(Ᵽ; Ọ; W) \& \rho_m(Ᵽ; Ọ; W)$ are non- symmetric weighted divergence measures.

4.Some new relations among divergences

We find a number of new significant and intriguing relationships between the new weighted divergence measures (3.1), (3.3), and (3.6) when compared to other common weighted divergence measures.

Proposition 4.1 Let Ᵽ, Ọ $\in \Gamma_n$, then the new intra relation shown below. following new intra relation. $\Upsilon_m(Ᵽ; Ọ; W) \leq \eta_m(Ᵽ; Ọ; W) \leq \rho_m(Ᵽ; Ọ; W)$, where m = 1,2,3... (4.1)

Proof: Because

$$\frac{w(\varepsilon^2-1)^{2m}[\varepsilon^4-\varepsilon^2+1]}{\varepsilon^{2m+1}} = \frac{w(\varepsilon^2-1)^{2m}}{\varepsilon^{2m-1}} + \frac{w(\varepsilon^2-1)^{2m+2}}{\varepsilon^{2m+1}}$$

&

$$\frac{w(\varepsilon^2-1)^{2m}}{\varepsilon^{2m-1}} e^{\frac{(\varepsilon^2-1)^2}{\varepsilon^2}} = \frac{w(\varepsilon^2-1)^{2m}}{\varepsilon^{2m-1}}\left[1 + \frac{(\varepsilon^2-1)^2}{\varepsilon^2} + \frac{(\varepsilon^2-1)^4}{2!\varepsilon^4} \ldots\right]$$

We obtain the following inequalities for m = 1, 2, 3….,& ε > 0, w> 0.

$$\frac{w(\varepsilon^2-1)^{2m}}{\varepsilon^{2m-1}} \leq \frac{w(\varepsilon^2-1)^{2m}}{\varepsilon^{2m-1}} + \frac{w(\varepsilon^2-1)^{2m+2}}{\varepsilon^{2m+1}} \leq \frac{w(\varepsilon^2-1)^{2m}}{\varepsilon^{2m-1}}\left[1 + \frac{(\varepsilon^2-1)^2}{\varepsilon^2} + \frac{(\varepsilon^2-1)^4}{2!\varepsilon^4} \ldots\right] \quad (4.2)$$

Put $\varepsilon = \frac{ƥ_i}{q_i}$, i = 1to n in (4.2) multiply by qi, then add up all the i = 1to n values to find the relation (4.1).

In particular, we will get the following from (4.1).

$\Upsilon_1(Ᵽ; Ọ; W) \leq \eta_1(Ᵽ; Ọ; W) \leq \rho_1(Ᵽ; Ọ; W)$

$\Upsilon_2(Ᵽ; Ọ; W) \leq \eta_2(Ᵽ; Ọ; W) \leq \rho_2(Ᵽ; Ọ; W)\ldots\ldots$ (4.3)

Remark Let ε ∈ (0, ∞), w> 0 and m = 1, 2, 3 lead to the new inequality that follows.

$$\frac{w(\varepsilon^2-1)^{2m}}{\varepsilon^{2m-1}} > \frac{w(\varepsilon-1)^{2m}}{\varepsilon^{\frac{2m-1}{2}}} \quad (4.4)$$

$$\frac{w(\varepsilon^2-1)^{2m}}{\varepsilon^{2m-1}} > \frac{w(\varepsilon-1)^{2m}}{(\varepsilon+1)^{2m-1}} \quad (4.5)$$

$$\frac{w(\varepsilon^2-1)^{2m}}{\varepsilon^{2m-1}} > w(\varepsilon-1)^{2m} \quad (4.6)$$

$$\frac{w(\varepsilon^2-1)^{2m}}{\varepsilon^{2m-1}} e^{\frac{(\varepsilon^2-1)^2}{\varepsilon^2}} > \frac{w(\varepsilon-1)^{2m}}{\varepsilon^{\frac{2m-1}{2}}} e^{\frac{(\varepsilon-1)^2}{\varepsilon}} \quad (4.7)$$

Every function used in (4.4) – (4.7) is convex and normalized, because $f''(\varepsilon, w) \geq 0 \ \forall \ \varepsilon > 0$, $w>0$ and $f(1,1) = 0$ respectively. Proof of these inequalities can be seen in "K. C. Jain and Praphull Chhabra [20]".

Proposition 4.2 Let Ᵽ, Ọ $\in \Gamma_n$ then we have the followings new inter relations

$$\Upsilon_m(Ᵽ;Ọ;W) > E^*_m(Ᵽ;Ọ;W) \quad (4.8)$$

$$\Upsilon_m(Ᵽ;Ọ;W) > \Delta_m(Ᵽ;Ọ;W) \quad (4.9)$$

$$\Upsilon_m(Ᵽ;Ọ;W) > \chi^{2m}(Ᵽ;Ọ;W) \quad (4.10)$$

$$\rho_m(Ᵽ;Ọ;W) > J^*_m(Ᵽ;Ọ;W) \quad (4.11)$$

Where
$\Upsilon_m(Ᵽ;Ọ;W)$, $E^*_m(Ᵽ;Ọ;W)$, $\Delta_m(Ᵽ;Ọ;W)$, $\chi^{2m}(Ᵽ;Ọ;W)$, $\rho_m(Ᵽ;Ọ;W)$ & $J^*_m(Ᵽ;Ọ;W)$ are given by (3.1), (1.2), (1.6), (1.7), (3.6) & (1.3) respectively.

Proof: Put $\varepsilon = \frac{p_i}{q_i}$, i = 1to n in (4.4) to (4.7), multiply by qi , then add up all i = 1to n, we get the relation (4.8) to (4.11) respectively.

Now, from (4.8) to (4.11), it is simple to state that

$$\Upsilon_1(Ᵽ;Ọ;W) > E^*_1(Ᵽ;Ọ;W) = E^*(Ᵽ;Ọ;W), \Upsilon_2(Ᵽ;Ọ;W) > E^*_2(Ᵽ;Ọ;W), \ldots\ldots \quad (4.12)$$

$$\Upsilon_1(Ᵽ;Ọ;W) > \Delta_1(Ᵽ;Ọ;W) = \Delta(Ᵽ;Ọ;W), \Upsilon_2(Ᵽ;Ọ;W) > \Delta_2(Ᵽ;Ọ;W), \ldots \quad (4.13)$$

$$\Upsilon_1(Ᵽ;Ọ;W) > \chi^2(Ᵽ;Ọ;W), \Upsilon_2(Ᵽ;Ọ;W) > \chi^4(Ᵽ;Ọ;W), \ldots\ldots \quad (4.14)$$

&

$$\rho_1(Ᵽ;Ọ;W) > J^*_1(Ᵽ;Ọ;W), \rho_2(Ᵽ;Ọ;W) > J^*_2(Ᵽ;Ọ;W), \ldots\ldots \quad (4.15)$$

Respectively.

Proposition 4.3 Let Ᵽ, Ọ ∈ Γ_n, then given below is new relations between divergences defined above.

$$\rho_m(Ᵽ;Ọ;W) > J^*_m(Ᵽ;Ọ;W) \geq E^*_m(Ᵽ;Ọ;W) \quad (4.16)$$

$$\rho_1(Ᵽ;Ọ;W) > 2\Delta(Ᵽ;Ọ;W) \geq 2\left[N^*_1(Ᵽ;Ọ;W) - N^*_2(Ᵽ;Ọ;W)\right] \quad (4.17)$$

$$\rho_1(Ᵽ;Ọ;W) > 8T(Ᵽ;Ọ;W) \geq J(Ᵽ;Ọ;W) \geq 8h(Ᵽ;Ọ;W) \geq 8I(Ᵽ;Ọ;W) \quad (4.18)$$

$$\rho_1(Ᵽ;Ọ;W) > 8A(Ᵽ;Ọ;W) \geq 8N_2(Ᵽ;Ọ;W) \geq 8N_3(Ᵽ;Ọ;W) \geq 8N_1(Ᵽ;Ọ;W) \geq 8L^*(Ᵽ;Ọ;W) \geq 8G^*(Ᵽ;Ọ;W) \geq 8H^*(Ᵽ;Ọ;W) \quad (4.19)$$

Proof: Since we know the followings.

$$J^*_m(Ᵽ;Ọ;W) \geq E^*_m(Ᵽ;Ọ;W) \quad [17] \quad (4.20)$$

$$\frac{1}{2}E^*(Ᵽ;Ọ;W) \geq \Delta(Ᵽ;Ọ;W) \geq \left[N^*_1(Ᵽ;Ọ;W) - N^*_2(Ᵽ;Ọ;W)\right] \quad [17] \quad (4.21)$$

$$\frac{1}{2}E^*(Ᵽ;Ọ;W) \geq T(Ᵽ;Ọ;W) \geq \frac{1}{8}J(Ᵽ;Ọ;W) \geq h(Ᵽ;Ọ;W) \geq I(Ᵽ;Ọ;W) \quad [24] \quad (4.22)$$

$$T(Ᵽ;Ọ;W) \geq A(Ᵽ;Ọ;W) \quad [17] \quad (4.23)$$

&

$$A(Ᵽ;Ọ;W)) \geq N_2(Ᵽ;Ọ;W) \geq N_3(Ᵽ;Ọ;W) \geq N_1(Ᵽ;Ọ;W) \geq L^*(Ᵽ;Ọ;W) \geq G^*(Ᵽ;Ọ;W) \geq H^*(Ᵽ;Ọ;W) \quad [42] \quad (4.24)$$

Combining (4.11) and (4.20), we obtain (4.16). We obtain (4.17) by combining the first and third parts of the established (4.16) at m = 1 with (4.21). We obtain (4.18) by combining the first and third parts of the established (4.16) at m = 1 with (4.22). We obtain (4.19) by combining the first and second parts of the established (4.18) with (4.23) & (4.24).

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